ISRAEL JOURNAL OF MATHEMATICS 75 (1991), 243–255

# LOWER BOUNDS FOR THE NUMBER OF CONJUGACY CLASSES IN FINITE SOLVABLE GROUPS

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ABSTRACT

We prove first that if G is a finite solvable group of derived length $d \geq 2$, then $k(G) > |G|^{1/(2^d-1)}$, where k(G) is the number of conjugacy classes in G. Next, a growth assumption on the sequence $[G^{(i)} : G^{(i+1)}]_1^{d-1}$, where $G^{(i)}$ is the *i*th derived group, leads to a $|G|^{1/(2d-1)}$ lower bound for k(G), from which we derive a $|G|^{c/\log_2\log_2|G|}$ lower bound, independent of d(G). Finally, "almost logarithmic" lower bounds are found for solvable groups with a nilpotent maximal subgroup, and for all Frobenius groups, solvable or not.

## 1. Introduction

In 1903 E. Landau [12] showed that the number $k(G)$ of conjugacy classes in a finite group $G$ cannot remain bounded as $|G| \to \infty$ . Using Landau's method, P. Erdös and P. Turán [6] gave the best general lower bound presently known: $k(G) > \log_2\log_2|G|$. But there is growing evidence that this bound is far from best possible. For example, the author proved in [1] that given $\epsilon > 0$, for almost all integers $n \leq x$, as $x \to \infty$, $k(G) > |G|^{1-\epsilon}$ for each group $G$ of order $n$. The recent work of A. V. López and J. V. López [13], [14] expands the classification of the finite groups with a given number $k$ of conjugacy classes to all $k \leq 12$, and shows that $k(G) > \log_3|G|$ when $|G| \leq 3^{13}$. No group $G$ has been discovered with $k(G) < \log_3|G|$, while $k(G) = \lceil\log_3|G|\rceil$ for either $G = \mathrm{PSL}(3,4)$ or $G = M_{22}$ (a Mathieu group), and 14 of the 26 sporadic simple groups satisfy $k(G) < \log_2|G|$.

On the other hand we know that if $G$ is a nilpotent group then $k(G) > \log_2|G|$ (see e.g. [3] or [18]). Furthermore, Kovács and Leedham-Green [11]

Received February 15, 1990 and in revised form September 13, 1990

have discovered, for each odd prime $p$, a group $G_p$ of order $p^p$ with fewer than $(\log_2 |G_p|)^3$ classes. We note that these $p$-groups $G_p$ contain far fewer classes relative to their orders than do the Alternating groups $A_n$ for $n$ sufficiently large. Cartwright proved in [3] that if $G$ is supersolvable then $k(G) \geq 3/5 \log_2 |G|$, and has shown more recently [4] that there exists a constant $a > 0$ such that for every solvable group $G$ we have $k(G) > a(\log_2 |G|)^b$, where $b \doteq .003459\ldots$.

Sherman proved [18] that if $G$ is nilpotent of nilpotence class $c$ then $k(G) \geq c(|G|^{1/c}-1)+1$, and Cartwright [3] showed that if $G$ is solvable of Fitting length $f$ then $k(G) > (\log_2 |G|)^{1/f}$.

In Theorem 1 we prove that if $G$ is a solvable group of derived length $d \geq 2$ then $k(G) > |G|^{1/2^d-1}$. This extends the author's result in [2] for $d = 2$, where it also shown that the exponent $1/3$ is best possible. There are many interesting situations which give rise to small derived lengths (e.g. $\leq 3$) while $|G| \to \infty$, assuming further knowledge of either the structure of solvable $G$ or the prime factorization of $|G|$, and these yield $|G|^\beta$ lower bounds on $k(G)$ via Theorem 1.

In Theorem 2 we show that certain growth assumptions on the sequence $\{[G^{(i)} : G^{(i+1)}]\}_1^{d-1}$ lead to a $|G|^{1/(2d-1)}$ lower bound on $k(G)$, which in turn leads to a $|G|^{c/\log_2 \log_2 |G|}$ lower bound independent of $d(G)$, using a deep result of P. M. Neumann and M. R. Vaughan-Lee. When $G$ is solvable and contains a nilpotent maximal subgroup we find in Theorem 3 that $k(G) \geq \frac{3}{8}(\log_2 |G|)^b$, where $b > .414\ldots$. Finally, we prove in Theorem 4 that for all Frobenius groups, solvable or not,

$$k(G) > \frac{1}{8}\frac{\log_2 |G|}{\log_2 \log_2 |G|}.$$

## 2.

$G$ is called a **Frobenius** group if $G$ contains a proper subgroup $H$ such that $H \cap H^g = \{1\}$ for every $g \in G - H$. Frobenius proved [8, p. 495] that $G - \bigcup_{g\in G} H^g \cup \{1\}$ forms a proper normal subgroup $N$ (the kernel). The studies of A. V. López and J. V. López [13], [14] show that for $k \leq 12$ either the largest or second largest solvable group containing exactly $k$ classes is a Frobenius group. Frobenius groups are extremal also in the sense of Lemma 1. For a different proof, without the characterization of equality, see [3, Lemma 2.1].

LEMMA 1: *If $N \trianglelefteq G$, then*

$$k(G) \geq k(G/N) + \frac{k(N)-1}{[G:N]}.$$

*When $N$ is proper, equality occurs if and only if $G$ is a Frobenius group with kernel $N$.*

*Proof:* Since $N \trianglelefteq G$, $k(G) = k_G(N) + k_G(G-N)$ where $k_G(S)$ is the number of $G$-conjugacy classes in the normal subset $S$. Clearly $k_G(G-N) \geq k(G/N) - 1$, since whenever $g_1, g_2 \in G - N$ and $g_1, g_2$ are $G$-conjugates then $Ng_1$ and $Ng_2$ are $G/N$-conjugates. Also, if $n \in N$ then $|C_G(n)N| = [C_G(n) : C_N(n)]|N|$ that is $[G : C_G(n)N]|[n]_N| = |[n]_G|$, where $[n]_N$ is the $N$-class of $n$ and $C_N(n) = N \cap C_G(n)$. It follows that each $G$-class $[n]_G \subseteq N$ splits into exactly $[G : C_G(n)N]$ $N$-classes. Summing over the distinct $[n]_G \subseteq N - \{1\}$ we obtain

$$k(N) - 1 = k_N(N - \{1\}) = \Sigma[G : C_G(n)N] \leq \Sigma[G : N] = [G : N](k_G(N) - 1),$$

so

$$k_G(N) \geq \frac{k(N) - 1}{[G : N]} + 1;$$

this together with $k_G(G - N) \geq k(G/N) - 1$ gives the desired inequality. It is well known that if $G$ is a Frobenius group with kernel $N$ then equality occurs in the statement of Lemma 1 [5, p. 68]. Suppose equality occurs and $1 \neq N \neq G$. From the proof we see that $[G : C_G(n)N] = [G : N]$, i.e., that $C_G(n) \subseteq N$ for each $n \in N - \{1\}$. But this is a necessary and sufficient condition that $G$ be a Frobenius group with kernel $N$ [10, p. 99]. □

THEOREM 1: *Suppose $G$ is a solvable group of derived length $d \geq 2$. Then $k(G) > |G|^{1/2^d - 1}$.*

*Proof:* More generally, we prove that ($G^{(i)}$ denotes the $i$th derived group) if $|G^{(i)}| \leq |G|^{1-\alpha}$ for some $i : 1 \leq i \leq d$ and $\alpha : 0 < \alpha \leq 1$, then $k(G) > |G|^{\alpha/2^i - 1}$. Since $k(G) > [G : G']$, the statement certainly follows when $i = 1$ and $\alpha > 0$. Assume, for a proof by induction, that for some $i : 1 \leq i \leq d - 1$, whenever $0 < \beta \leq 1$ and $[G : G^{(i)}] \geq |G|^\beta$, then $k(G) > |G|^{\beta/2^i - 1}$. Now suppose that $[G : G^{(i+1)}] \geq |G|^\alpha$ for some $\alpha : 0 < \alpha \leq 1$. Then either (a) or (b) must hold, where:

(a) $$[G : G^{(i)}] \geq |G|^{\left(\frac{2^i - 1}{2^{i+1} - 1}\right)\alpha},$$

(b) $$[G^{(i)} : G^{(i+1)}] \geq |G|^{\left(\frac{2^{i+1} - 2^i}{2^{i+1} - 1}\right)\alpha},$$

If (a) is true, then the induction hypothesis with

$$\beta = \left(\frac{2^i - 1}{2^{i+1} - 1}\right)\alpha$$

yields $k(G) > |G|^{\beta/2^i - 1} = |G|^{\alpha/2^{i+1}-1}$, so the conclusion holds for $i+1$. Otherwise (b) is true and (a) is false. From Lemma 1 and $k(G^{(i)}) > [G^{(i)} : G^{(i+1)}]$ we have

$$k(G) \geq k(G/G^{(i)}) + \frac{k(G^{(i)})-1}{[G:G^{(i)}]} > \frac{[G^{(i)}:G^{(i+1)}]}{[G:G^{(i)}]}$$

which is now $> |G|^{\alpha/2^{i+1}-1}$. In each case we find that $[G:G^{(i+1)}] \geq |G|^{\alpha}$ implies $k(G) > |G|^{\alpha/2^{i+1}-1}$, and the induction is complete. □

*Remarks:* P. M. Neumann and M. R. Vaughan-Lee showed (in [16]) that there exist positive constants $a < 8$ and $b < 8/3$ such that, if $G$ is any finite solvable group, then $d(G) < a + b\log_2\log_2|G|$. Our Theorem 1 shows that whenever solvable $G$ has $d(G) \leq \log_2\log_2|G| - \log_2\log_2\log_2|G|$ then $k(G) > \log_2|G|$. There are many interesting situations where the derived length of solvable $G$ is known to be small (e.g. $\leq 3$), assuming a little more knowledge about $G$, or $|G|$. These in turn lead to $|G|^{\beta}$ lower bounds for $k(G)$, by Theorem 1. For example, if $G$ is solvable of cube-free order then $d(G) \leq 3$ (see [19]). If $G$ has an abelian maximal subgroup then $G$ is solvable and $d(G) \leq 3$ ([17], p. 392). If $G = AB$ where $A$ and $B$ are abelian subgroups of $G$ then $G$ is solvable and $d(G) \leq 2$ ([17], p. 384). Finally, if $G$ is a solvable, doubly transitive permutation group then it follows from the work of Huppert [9] that $d(G) \leq 5$.

LEMMA 2: *If $G$ is a solvable group and $N \trianglelefteq G$ then $k(G) \geq k(G/N) + d(N)$, where $d(N)$ is the derived length of $N$. In particular, $k(G) \geq [G:G'] + d(G) - 1$.*

*Proof:* The inequality in the lemma is certainly true when $N$ is abelian. Let $d = d(N)$, and $N > N^{(1)} > \cdots > N^{(d)} = \{1\}$ the derived series for $N$. Then each $N^{(i)}$ is characteristic in $N$ and normal in $G$, and $N^{(i)}/N^{(i+1)}$ is abelian. Therefore for each $i \leq d-1$ we have $k(G/N^{(i+1)}) \geq k(G/N^{(i)}) + 1$. Thus

$$k(G) \geq k(G/N^{(d-1)}) + 1 \geq k(G/N^{(d-2)}) + 2 \geq \cdots \geq k(G/N) + d(N).$$

□

A sequence of real numbers $\{a_i\}_1^n$ "decreases on the average" if

$$a_{r+1} \leq \frac{1}{r}(a_1 + a_2 + \cdots + a_r) \quad \text{for each } r : 1 \leq r \leq n-1.$$

Note that this condition is equivalent to

$$\frac{1}{r+1}(a_1 + a_2 + \cdots + a_{r+1}) \leq \frac{1}{r}(a_1 + a_2 + \cdots + a_r).$$

Clearly, non-increasing sequences are decreasing on the average, but it is easy to construct others. An interesting inequality, generalizing a classical inequality of Chebycheff, states that if the sequences $\{a_i\}_1^n$ and $\{b_i\}_1^n$ both decrease on the average (or both increase on the average), then

$$n \sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i.$$

See e.g. [15]. A proof by induction is tricky, but not difficult. This inequality is used to prove the following result.

THEOREM 2: *Suppose $G$ is a solvable group of derived length $d \geq 3$, and that $\{[G^{(i)} : G^{(i+1)}]\}_1^{d-1}$ decreases on the average. Then*
(a) $k(G) > |G|^{1/(2d-1)}$, *and*
(b) $k(G) > |G|^{c/\log_2 \log_2 |G|}$ *where $c$ is a positive constant.*

*Proof:* In Lemma 1 replace $G$ by $G^{(i)}$ and $N$ by $G^{(i+1)}$. Then

$$k(G^{(i)}) \geq [G^{(i)} : G^{(i+1)}] + \frac{k(G^{(i+1)}) - 1}{[G^{(i)} : G^{(i+1)}]},$$

so

$$\frac{k(G^{(i)}) - 1}{[G : G^{(i)}]} \geq \frac{[G^{(i)} : G^{(i+1)}] - 1}{[G : G^{(i)}]} + \frac{k(G^{(i+1)}) - 1}{[G : G^{(i+1)}]} \quad \text{for each } i : 1 \leq i \leq d-2.$$

We may now use the latter inequality to repeatedly replace $(k(G^{(i)})-1)/[G : G^{(i)}]$ by its lower bound. Thus,

$$\begin{aligned} k(G) &\geq [G : G'] + \frac{k(G') - 1}{[G : G']} \geq [G : G'] + \frac{[G' : G''] - 1}{[G : G']} + \frac{k(G'') - 1}{[G : G'']} \\ &= [G : G'] + \frac{[G' : G''](\mid G' \mid - \mid G'' \mid)}{\mid G \mid} + \frac{k(G'') - 1}{[G : G'']} \geq \cdots \geq, \end{aligned}$$

so

$$(1) \qquad k(G) \geq [G : G'] + \frac{1}{|G|} \sum_{i=1}^{d-1} [G^{(i)} : G^{(i+1)}](|G^{(i)}| - |G^{(i+1)}|).$$

Since $\{[G^{(i)} : G^{(i+1)}]\}$ decreases on the average, so does $\{[G^{(i)} : G^{(i+1)}] - 1\}$, and the sequence $\{|G^{(i)}| - |G^{(i+1)}|\}$ must decrease since $|G^{(i+1)}|$ properly divides

$|G^{(i)}|$. From the generalized Chebycheff inequality we have (after rewriting the telescoping sum)

$$(*)\quad \sum_{i=1}^{d-1}[G^{(i)}:G^{(i+1)}](|G^{(i)}|-|G^{(i+1)}|) \geq \left(\frac{1}{d-1}\sum_{i=1}^{d-1}[G^{(i)}:G^{(i+1)}]\right)(|G'|-1).$$

Since the arithmetic-geometric inequality gives (after rewriting the telescoping product):

$$\frac{1}{d-1}\sum_{i=1}^{d-1}[G^{(i)}:G^{(i+1)}] \geq |G'|^{1/(d-1)},$$

we obtain from (1) and (*) that

$$k(G) \geq [G:G'] + \frac{1-|G'|^{-1}}{[G:G']}|G'|^{1/(d-1)}.$$

From the latter (or Lemma 1 or 2), part (a) of the Theorem follows when

$$[G:G'] \geq |G|^{1/(2d-1)}.$$

On the other hand,

$$[G:G'] < |G|^{1/(2d-1)} \quad \text{iff } |G'|^{1/(d-1)} > |G|^{2/(2d-1)}.$$

The latter, together with the last inequality on $k(G)$ and the arithmetic-geometric inequality, again yield part (a) of the Theorem. As remarked earlier, P. M. Neumann and M. R. Vaughan-Lee showed (Theorem 10.2 in [16]) that there exist positive constants $a < 8$, $b < 8/3$ such that if $G$ is any finite solvable group then $d(G) < a + b\log_2\log_2|G|$. Part (b) of the theorem follows from the latter inequality and part (a). □

From Lemma 1 it follows that if $G$ is solvable of derived length $d$, and

$$|G^{(d-1)}| \geq |G|^{(1+\alpha)/2}$$

for some $\alpha : 0 \leq \alpha \leq 1$, then $k(G) \geq |G|^{\alpha}$. Inequality (1) in the proof of Theorem 2 gives us a little more from the same lower bound on $\max_{1\leq j\leq d-2}[G^{(j)}:G^{(j+1)}]$.

COROLLARY: *For a solvable group $G$, suppose*

$$\max_{1\le j\le d-2}[G^{(j)}:G^{(j+1)}]\ge |G|^{(1+\alpha)/2}.$$

*Then $k(G)\ge [G:G']+|G|^{\alpha}$. In particular, if*

$$\max_{1\le j\le d-2}[G^{(j)}:G^{(j+1)}]\ge (|G|\log_2|G|)^{1/2}$$

*then $k(G)\ge [G:G']+\log_2|G|$.*

*Proof:* Whenever $x\ge 2y\ge 4$ we have $x-y\ge x/y$, so $j\le d-2$ implies

$$|G^{(j)}|-|G^{(j+1)}|\ge [G^{(j)}:G^{(j+1)}],$$

and now by hypothesis at least one term in $\sum_{i=1}^{d-1}[G^{(i)}:G^{(i+1)}](|G^{(i)}|-|G^{(i+1)}|)$ is at least $|G|^{1+\alpha}$. The first conclusion now follows from inequality (1) in the proof of Theorem 2. In particular when

$$\alpha=\frac{\log_2\log_2|G|}{\log_2|G|},$$

the second conclusion is obtained. □

Suppose the solvable group $G$ contains a nilpotent maximal subgroup $M$, and the Fitting subgroup $F(G)\not\le M$. Then $G=FM$ and hence $G$ has Fitting length 2. It follows from Theorem 4 of [3] that $k(G)>(\log_2|G|)^{1/2}$. The general case is considered in Theorem 3 below, but we first need the following Lemma.

LEMMA 3: *Let $N$ be a proper normal subgroup of $G$, and $0<\alpha,\beta,a,b\le 1$.*
*(i) If $k(N)\ge |N|^{\alpha}$ and $k(G/N)\ge [G:N]^{\beta}$ then*

$$k(G)>|G|^{\alpha\beta/(1+\alpha+\beta)}.$$

*(ii) If $k(N)\ge a|N|^{\alpha}$ and $k(G/N)\ge b\log_2[G:N]$ then*

$$k(G)>\frac{ab}{1+\alpha}(\alpha\log_2|G|-\log_2\log_2|G|).$$

*Proof:* (i) If $|N|\ge |G|^{(1+\beta)/(1+\alpha+\beta)}$, then $|N|^{1+\alpha}\ge |G|^{1+\alpha\beta/(1+\alpha+\beta)}$. By Lemma 1

$$k(G)\ge k(G/N)+\frac{|N|^{\alpha}-1}{[G:N]}>\frac{|N|^{\alpha+1}}{|G|}\ge |G|^{\alpha\beta/(1+\alpha+\beta)}.$$

Otherwise $|N| < |G|^{(1+\beta)/(1+\alpha+\beta)}$, i.e. $[G:N] > |G|^{\alpha/(1+\alpha+\beta)}$, and by Lemma 1,

$$k(G) > k(G/N) \geq [G:N]^{\beta} > |G|^{\alpha\beta/(1+\alpha+\beta)}.$$

The proof of (ii) is similar, according to whether

$$|N|^{1+\alpha} > \frac{b}{1+\alpha}|G|\log_2|G|,$$

or

$$|N|^{1+\alpha} \leq \frac{b}{1+\alpha}|G|\log_2|G|.$$

In the former case Lemma 1 gives

$$k(G) > \frac{k(N)}{[G:N]} \geq \frac{a|N|^{\alpha}}{[G:N]} > \frac{ab}{1+\alpha}\log_2|G|.$$

In the latter case Lemma 1 gives

$$k(G) > k(G/N) > \frac{b}{1+\alpha}(\alpha\log_2|G| - \log_2\log_2|G|).$$

□

THEOREM 3: *Let $G$ be a solvable group with a nilpotent maximal subgroup. Then*

(a) $k(G) > \frac{3}{8}(\log_2|G|)^b$, *where $b$ ($> .414$) satisfies $b^{-1} = 1 + \log_2(8/3)$.*

(b) *If $|G| = \prod p_i^{\alpha_i}$ ($p_i$ distinct primes) and $s = \mathrm{Max}\ \alpha_i$ then*

$$k(G) > \frac{2}{5}\frac{\log_2|G|}{s+1}.$$

*Proof:* Let $M$ be a nilpotent maximal subgroup of $G$ and $M_G$ the core of $M$. In order to prove (a) we will need $k(G/M_G) > 3/4\log_2[G:M_G]$, and for (b) we need $k(G/\Phi) > 1/4\log_2[G:\Phi]$, where $\Phi(G)$ is the Frattini subgroup of $G$. These two inequalities are derived as follows: Since $G$ is solvable, if $L/M_G$ is a chief factor of $G$ then $L/M_G$ is abelian and $L \trianglelefteq G$, but $L \not\leq M$. Also $L' \leq M_G \leq L \cap M$ so $L \cap M \trianglelefteq L$, $M$ and $M_G \leq L \cap M \trianglelefteq G = LM$. Thus $M_G = L \cap M$ so $M/M_G \cong G/L \cong G/M_G/L/M_G$, and $G/M_G$ is abelian by nilpotent. By Theorem 1 of [3], $k(G/M_G) \geq 3/4\log_2[G:M_G]$. Since $M_G$ is nilpotent and normal in $G$, $M_G \leq F(G)$, the Fitting subgroup of $G$. Also $\Phi(G) \leq M_G$, $M_G/\Phi$ is abelian and $G/\Phi/M_G/\Phi \cong G/M_G$. In Lemma 3(ii) replace $G$ by $G/\Phi$, $N$ by $M_G/\Phi$ and set $\alpha = a = 1$, $b = 3/4$. Then

$$k(G/\Phi) > \tfrac{3}{8}(\log_2[G:\Phi] - \log_2\log_2[G:\Phi]).$$

The latter is $\geq 1/4\log_2[G:\Phi]$ when $[G:\Phi] > 2^{10}$, so $k(G/\Phi) > 1/4\log_2[G:\Phi]$. But all groups of order $t \leq 2^{10}$ have $> 1/4\log_2 t$ conjugacy classes [13], so in all cases $k(G/\Phi) > \frac{1}{4}\log_2[G:\Phi]$.

*Proof of* (a): For $0 \leq j \leq d(M_G)$ define $G_j = G/M_G^{(j)}$ and the abelian $N_j = M_G^{(j-1)}/M_G^{(j)}$ for $j \geq 1$. Then $G_j/N_j \cong G_{j-1}$ and we prove by induction that $k(G_j) > (\frac{3}{8})^{j+1}\log_2|G_j|$, $j \geq 0$. For $j = 0$ we have

$$k(G_o) = k(G/M_G) > \frac{3}{4}\log_2[G : M_G],$$

as proved in the paragraph above. Thus $k(G_o) > \frac{3}{4}\log_2|G_o|$. When $j \geq 1, N_j$ is abelian and we apply Lemma 3(ii) to $k(G_j)$, with $N$ replaced by $N_j$ and $a = \alpha = 1$. For $j = 1$ set $b = 3/4$, and for $j \geq 2$ set $b = (3/8)^j$. Then

$$k(G_1) > 3/8(\log_2|G_1| - \log_2\log_2|G_1|),$$

and for $j \geq 2$

$$k(G_j) > (\frac{3}{8})^j\frac{1}{2}(\log_2|G_j| - \log_2\log_2|G_j|).$$

Since we may assume that $|G_1| \geq 2^7$ (otherwise $(3/8)^2\log_2|G_1| < 1 \leq k(G_1)$), we know that $\log_2|G_1| > 8/5\log_2\log_2|G_1|$ so

$$\log_2|G_1| - \log_2\log_2|G_1| > 3/8\log_2|G_1|$$

and the case $j = 1$ follows. For $j \geq 2$ we may assume that $|G_j| \geq 2^{16}$, in which case $\log_2|G_j| \geq 4\log_2\log_2|G_j|$ and

$$k(G_j) > (\tfrac{3}{8})^j(\tfrac{1}{2})(\tfrac{3}{4}\log_2|G_j|) = (\tfrac{3}{8})^{j+1}\log_2|G_j|.$$

In particular, when $j = d(M_G)$ we find that $k(G) > (3/8)^{d(M_G)+1}\log_2|G|$.

Set $b^{-1} = 1 + \log_2(8/3)$. As usual, we have $2^{d(M_G)-1} \leq c$, where $c$ is the nilpotence class of the nilpotent group $M_G$. And now we obtain, using Lemma 6.1(i) of [3], that $k(G) > c \geq 2^{d(M_G)-1}$. If we assume that $d(M_G) \geq b\log_2\log_2|G|$, then $k(G) > 1/2(\log_2|G|)^b$ as desired. On the other hand, suppose that $d(M_G) \leq b\log_2\log_2|G|$. From the definition of $b$ we have

$$b\log_2\log_2|G| = \frac{b\log_{8/3}\log_2|G|}{\log_{8/3}2} = (1-b)\log_{8/3}\log_2|G|.$$

The inequality in (a) now follows in this case also from

$$k(G) > (3/8)^{d(M_G)+1}\log_2|G|.$$

*Proof of (b):* Here we actually prove that

$$(*) \qquad k(G) > \frac{1}{2(s+1)}\log_2|G| - \frac{1}{4}\log_2\log_2|G|.$$

Since always $k(G) > \log_2 \log_2 |G|$ [6], (b) is immediate if

$$\log_2 \log_2 |G| \geq \frac{2}{5} \frac{\log_2 |G|}{s+1}.$$

On the other hand (b) follows from (*) when

$$\log_2 \log_2 |G| < \frac{2}{5} \frac{\log_2 |G|}{s+1}.$$

To prove (*), note that when $s \leq 2$ we have each Sylow subgroup of $G$ abelian, as well as $M$ an abelian maximal subgroup of $G$. As mentioned in the Remarks, either leads to $d(G) \leq 3$ and hence $k(G) > |G|^{1/7}$, by Theorem 1. Since $s \geq 1$ we need only show that $|G|^{1/7} > \frac{1}{5} \log_2 |G|$; the details are left to the reader. So assume that $s \geq 3$. By a result of Hill and Parker [7] the nilpotence class $\mathrm{cl}(\Phi(G)) \leq \frac{1}{2}(s-1)$, and by Sherman's theorem [18] $k(\Phi) > |\Phi|^{2/(s-1)}$. Recall now that $k(G/\Phi) > \frac{1}{4} \log_2 [G : \Phi]$, proved in the first paragraph. In Lemma 3(ii) replace $N$ by $\Phi$, $\alpha$ by $2/(s-1)$, and set $a = 1$ and $b = 1/4$. We obtain the inequality

$$k(G) > \frac{1}{2(s+1)}(\log_2 |G| - (\frac{s-1}{2}) \log_2 \log_2 |G|),$$

and from this (*) follows. □

LEMMA 4: *Let $N$ be a nilpotent normal subgroup of the solvable group $G$, with $k(G/N) \geq [G : N]^\beta$. Then*

$$k(G) > \frac{\beta}{2(\beta+1)} \log_2 |G| / \log_2 \log_2 |G|.$$

*Proof:* Let $\{N^{(j)} : j \geq 0\}$ denote the derived series of $N$, $N^{(0)} = N$, and $G_j := G/N^{(j)}$. If $N_j := N^{(j-1)}/N^{(j)}$ are the abelian factor groups of $\{N^{(j)}\}$, then $G_{j-1} \cong G_j/N_j$, for $j \geq 1$. Hence $k(G_o) = k(G/N) \geq [G : N]^\beta = |G_o|^\beta$, $k(G_1/N_1) = k(G_o) \geq [G_1 : N_1]^\beta$, and $k(N_1) = |N_1|$. By Lemma 3(i), $k(G_1) \geq |G_1|^{\beta_1}$, where $\beta_1 = \beta/(2+\beta)$. Now $k(G_2/N_2) \geq [G_2 : N_2]^{\beta_1}$ and $k(N_2) = |N_2|$, so $k(G_2) \geq |G_2|^{\beta_2}$ where $\beta_2 = \beta_1/(2+\beta_1) = \beta/(4+3\beta)$. Continuing in this way we see that $k(G_i) \geq |G_i|^{\beta_i}$ where

$$\beta_i = \frac{1}{(1+1/\beta)2^i - 1}, \quad \text{for each } i : 1 \leq i \leq d(N).$$

In particular, when $i = d(N)$, $G_i \cong G$, and $k(G) \geq |G|^\gamma$, where

$$\gamma^{-1} = (1+1/\beta)2^{d(N)} - 1.$$

Since the nilpotence class $c = \mathrm{cl}(N)$ always satisfies $d(N) \leq \log_2 \mathrm{cl}(N) + 1$ we have $\gamma^{-1} \leq 2(1 + 1/\beta)c(N) - 1$. By [3, Lemma 6.1(i)] we know that $k(G) \geq k(G/N) + c(N)$. If

$$c(N) > \frac{(\log_2 |G|/\log_2 \log_2 |G|) + 1}{2(1 + 1/\beta)}$$

we are done. On the other hand, suppose

$$2(1 + 1/\beta)c(N) \leq (\log_2 |G|/\log_2 \log_2 |G|) + 1.$$

Then $\gamma^{-1} \leq \log_2 |G|/\log_2 \log_2 |G|$, and $k(G) \geq |G|^\gamma \geq \log_2 |G|$, so again the desired conclusion follows. □

THEOREM 4: *If $G$ is a Frobenius group, then*

$$k(G) > \frac{1}{8} \frac{\log_2 |G|}{\log_2 \log_2 |G|}.$$

*Proof:* Let $N$ be the kernel of $G$, and $H$ the complement, with $G = HN$, $H \cap N = \{1\}$. We use many of the known properties of Frobenius groups (see e.g., [8, Chap. V, ξ 8].

If $|H|$ is odd then $H \cong G/N$ is metacyclic. In particular, $H'$ is abelian and thus, by Theorem 1, $k(G/N) \geq [G : N]^{1/3}$. Also the kernel $N$ is nilpotent, so by Lemma 4

$$k(G) > 1/8 \log_2 |G|/\log_2 \log_2 |G|.$$

On the other hand, if $|H|$ is even then $N$ is abelian. When $H$ is also solvable, then $H$ has a normal subgroup $H_o$ such that all Sylow subgroups of $H_o$ are cyclic, and $H/H_o$ is isomorphic to a subgroup of Sym(4). Thus $G/N$ has a normal subgroup $M/N$ such that $G/M$ is isomorphic to a subgroup of Sym(4). Since all subgroups $L$ of Sym(4) satisfy $k(L) \geq 5/24|L|$, we have

$$k(G) > k(G/M) \geq 5/24[G : M].$$

If

$$[G : M] > \frac{3}{5} \frac{\log_2 |G|}{\log_2 \log_2 |G|}$$

we are done, so assume that

$$|M| > \tfrac{5}{3}|G| \log_2 \log_2 |G|/\log_2 |G|.$$

Since all Sylow subgroups of $H_o \cong M/N$ are cyclic, $H_o'$ is cyclic and $M'' \leq N$, so $d(M) \leq 3$. By Theorem 1, $k(M) \geq |M|^{1/7}$, and using Lemma 1

$$k(G) > \frac{k(M)}{[G:M]} \geq \frac{|M|^{8/7}}{|G|}.$$

But now the lower bound assumed for $|M|$ easily yields

$$\frac{|M|^{8/7}}{|G|} > \frac{1}{8}\frac{\log_2 |G|}{\log_2 \log_2 |G|}.$$

Finally, when $|H|$ is even but $H$ is not solvable, $H$ has a normal subgroup $H_o$ such that $[H : H_o] \leq 2$ and $H_o \cong \mathrm{SL}(2,5) \times H_1$, where $H_1$ has only cyclic Sylow subgroups. We will show that $k(H) > |H|^{1/3}$, i.e. $k(G/N) > [G:N]^{1/3}$. Since $N$ is abelian, using Lemma 3(i) we then obtain

$$k(G) > |G|^{1/7} > \frac{1/8 \log_2 |G|}{\log_2 \log_2 |G|}.$$

Now $k(H_o) = k(\mathrm{SL}(2,5)) \cdot k(H_1) \geq 9(9/2)^{1/3}|H_1|^{1/3}$, since $k(\mathrm{SL}(2,5)) = 9$ and $H_1$ is metabelian. But $|H_1| = |H_o|/120 \geq |H|/240$, and

$$k(H) > \frac{k(H_o)}{[H:H_o]}$$

by Lemma 1. So

$$k(H) > 1/2k(H_o) \geq \left(\frac{9}{2}\right)^{4/3} \cdot \left(\frac{|H|}{240}\right)^{1/3} > |H|^{1/3},$$

and the proof is complete. □

*Note added in proof.* In a paper to appear in the *Journal of the London Mathematical Society*, L. Pyber proves that there exists an explicitly computable constant $\varepsilon > 0$ such that every group $G$ of order $n > 4$ satisfies $k(G) \geq \varepsilon \log n/(\log\log n)^8$. The proof relies on the classification of the finite simple groups. His estimate for solvable groups, which does not rely on the classification of the finite simple groups, is somewhat better but not as good as $k(G) \geq \log n/(\log\log n)$.

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